Matter waves in atomic-molecular BEC with Feshbach resonance management

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Abstract: Dynamics of matter waves in the atomic to molecular condensate transition with a time modulated atomic scattering length is investigated. The conditions for dynamical suppression of association of atoms into the molecular field are obtained.

1. Introduction

The dynamics of nonlinear waves in quadratic nonlinear media with periodic modulations in time of the parameters is an active area for investigations. The main type of modulation that attracts interest, is the case of periodically poling crystals with rapid modulations of the quadratic nonlinearity parameter along the direction of propagation [1] and the periodic modulation of mismatch parameters, which can be realized in a nonlinear optical media [3], or for matter waves in atomic-molecular condensates [4]. Represent of interest an investigation of dynamics of waves in quadratic nonlinear media with a Kerr nonlinearity varying along the direction of propagation. System described by this type of model is an atomic-molecular condensate, while varying in time the atomic scattering length,that can be implemented by the so-called Feshbach resonance management.

2. The model. Averaged $\chi^{(2)}$system

The system, describing the propagation of the fundamental- (FH) and second- (SH) harmonics in a quadratic nonlinear media with a cubic nonlinearity have in standard optics dimensionless variables the form [**?**]

$$\begin{aligned} iu_z + u_{xx} + \gamma(z)|u|^2 u + u^* v &= 0, \\ iv_z + \frac{1}{2} v_{xx} + qv + \frac{u^2}{2} &= 0. \end{aligned} \tag{1}$$

Here u, v are the fields of the FH and the SH respectively. For an atomic-molecular BEC system they are instead the atomic and molecular fields [5].

We here study the propagation of continuous waves (CW), i.e. when $u_{xx} = v_{xx} = 0$, in a media with a periodically varying Kerr nonlinearity.

We obtain the following system of averaged coupled equations for the FH and SH

$$\begin{aligned} i\bar{u}_z - \frac{\gamma_1}{\omega} J_1\left(\frac{2\gamma_1}{\omega}|\bar{u}|^2\right)\bar{u}(\bar{u}^2\bar{v}^* + \bar{u}^{*2}\bar{v}) + \bar{u}^*\bar{v}J_0\left(\frac{2\gamma_1}{\omega}|\bar{u}|^2\right) + \gamma_0|\bar{u}|^2\bar{u} = 0, \\ i\bar{v}_z + q\bar{v} + \frac{\bar{u}^2}{2} J_0\left(\frac{2\gamma_1}{\omega}|\bar{u}|^2\right) = 0. \end{aligned} \tag{2}$$

The Hamiltonian of the above averaged system, i.e.

$$i\bar{u}_z = -\frac{\partial H}{\partial \bar{u}^*}, i\bar{v}_z = -\frac{\partial H}{\partial \bar{v}^*},$$

is then

$$H = \frac{\gamma_0}{2}|\bar{u}|^4 + q|\bar{v}|^2 + \frac{1}{2}J_0\left(\frac{2\gamma_1}{\omega}|\bar{u}|^2\right)(\bar{u}^2\bar{v}^* + \bar{u}^{*2}\bar{v}). \tag{3}$$

From the above equation we conclude the important result that the Hamiltonian have the same form as the standard one, but with a renormalized effective quadratic nonlinearity parameter

$$\chi_{\text{eff}} = J_0\left(\frac{2\gamma_1}{\omega}|\bar{u}|^2\right)\chi. \tag{4}$$

Note that the renormalization depends nonlinearly on the intensity of the FH field. For atomic-molecular BEC system, it means that the renormalized atom-molecular interaction depend nonlinearly on the atomic population.

Here the modulations of the cubic nonlinearity can lead to a weakening of the effective quadratic interaction, i.e. Eq. (4). A new effect seen here is that in the condition for a vanishing effective coupling ($\chi_{\text{eff}} = 0$), which means a zero of the Bessel function, enters the intensity of the fundamental harmonic (atomic population). The comparison of results by numerical simulations of the full (1) and averaged equations (2) show a good agreement.

2.1. Dynamics for slow modulation

Let us study the dynamics of the system under slow resonant modulations. For small amplitude of oscillations we can write:

$$w(\xi) \approx w_3 a \sin^2(r\xi) = \frac{w_3 a}{2}(1 - \cos(2r\xi)). \tag{5}$$

To check the resonant behavior in the FH to SH oscillations, we take the frequency for the modulation of the mean-field cubic (Kerr) nonlinearity to be equal to the frequency of Eq. (5), i.e. $\omega = 2r$. The numerical integration shows an resonant enhancement for the amplitude of the oscillations for the second harmonic (the molecular field) generation, which is growing with γ_1. Chaotic oscillations originating from homoclinic crossing are also possible here. To investigate possible chaotic regimes of oscillations, we calculate the Melnikov function $M(z_0)$ for the particular case $\gamma_0 = 0$,

$$M(z_0) = -\frac{\pi\gamma_1}{3} \frac{\omega^2(2+\omega^2)}{\sinh(\frac{\pi\omega}{\sqrt{2}})} \sin(\omega z_0). \tag{6}$$

Since the Melnikov function above have infinite number of zeros, chaos in the harmonic generation(molecular field) is expected to occur.

In conclusion, we have obtained the parameters of modulations for which the SH generation (association of atoms into a molecular condensate) can be suppressed dynamically. For the case of slow modulations, we find an enhancement of the SH (i.e.the molecular condensate) generation for the resonant value of the frequency for the modulations of the nonlinearity, which for strong amplitudes are chaotic.

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Matter Waves in Atomic-Molecular BEC With Feshbach Resonance Management

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Introduction

The dynamics of nonlinear waves in quadratic nonlinear media with periodic modulations in time of the parameters is an active area for investigations [1]. The interest is connected with the possibility of quasi-phase-matching in such media and the ability to control the interactions of the waves. The main type of modulation that attracts interest, is the case of periodically poling crystals with rapid modulations of the quadratic nonlinearity parameter along the direction of propagation. Another interesting case is the periodic modulation of mismatch parameters, which can be realized in a nonlinear optical media, or for matter waves in atomic-molecular condensates.

Here we will investigate analytically and numerically the propagation of continuous waves in a quadratic nonlinear ($\chi^{(2)}$) system with an additional nonuniform cubic (Kerr) nonlinearity.

The Model

The system, describing the propagation of the fundamental- (FH) and second- (SH) harmonics in a quadratic nonlinear media with a cubic nonlinearity have in standard optics dimensionless variables the form [1]

$$iu_z + u_{xx} + \gamma(z)|u|^2u + u^*v = 0,$$
$$iv_z + \frac{1}{2}v_{xx} + qv + \frac{u^2}{2} = 0. \quad (1)$$

Here u, v are the fields of the FH and the SH respectively.

The related system of coupled Gross-Pitaevskii like equations with conversion terms describing the atomic-molecular BEC is in physical units

$$i\hbar\psi_{a,T} = -\frac{\hbar^2}{2m_a}\psi_{a,XX} + \sum_{j=a,m} g_{aj}|\psi_j|^2\psi_a + G_{am}\psi_a^*\psi_m,$$
$$i\hbar\psi_{m,T} = -\frac{\hbar^2}{2m_m}\psi_{m,XX} + \delta\omega\psi_m + \sum_{j=a,m} g_{jm}|\psi_j|^2\psi_m + G_{am}\frac{\psi_a^2}{2} \quad (2)$$

where $m_m = 2m_a$ are the masses, $\delta\omega$ is the energy detuning, g_{aa}, g_{mm}, g_{am} are one-dimensional parameters for the atom-atom, molecule-molecule, and the atom-molecular interactions [2], with $g_{aa} = 2\hbar\omega_\perp a_s$, where a_s is the atomic scattering length and the parameter G_{am} is the strength of the atom-molecule conversion, while effects of elastic collisions involving molecules, g_{mm} and g_{am} in Eq. (2), will be neglected here. The dimensionless form of Eqs. (1) is obtained by the following change of variables in the system (2) $t = T\omega_\perp, x = \frac{\sqrt{2}X}{l_a}, l_a = \sqrt{\frac{\hbar}{m_a\omega_\perp}}u = \sqrt{\frac{G_{am}}{\hbar\omega_\perp}}\psi_a, v = \frac{G_{am}}{\hbar\omega_\perp}\psi_m, \gamma = \frac{g_{aa}}{G_{am}}, q = \frac{\delta\omega}{\hbar\omega_\perp}$. Time dependent variations of the atomic scattering length a_s and therefor the parameter γ, can be obtained by the variations in time of an external magnetic field near a resonant value, so-called Feshbach resonance management technics [3].

In this work we will investigate continues waves ($u_{xx} = v_{xx} = 0$) with rapid and slow periodic variations of the cubic nonlinearity. The modulations are taken in the form: $\gamma = \gamma_0 + \gamma_1\cos(\omega z)$

Unperturbed $\chi^{(2)}$ system with Kerr nonlinearity

We apply conventional normalizations for the amplitudes, the direction of amplitudes, and the mismatch, as $u = \sqrt{I}\rho e^{i\phi}$, $v = \sqrt{I}\mu e^{i\psi}$, $Z = z/L$, $\kappa = qL/2$, where $I = |u|^2 + 2|v|^2$ is the conserved total intensity (i.e., $\rho^2 + 2\mu^2 = 1$). For the atomic-molecular BEC system it have the interpretation of the total number of particles. By defining the parameters, $\beta = L\sqrt{I}$, $\Upsilon = LI\gamma_0$, we can obtain differential equation for the relative intensity, $w = \mu^2$, of the second harmonic

$$\dot{w}^2 + P(w) = 0, \quad (3)$$

which is equivalent to the dynamical equation describing a classical particle moving in a potential $P(w)$. The potential $P(w)$ is in the form of a quartic polynomial

$$P(w) = D^2w^4 - 4(1+DC)w^3 + 4(1 - \frac{1}{2}HD + C^2)w^2 - (1 - 4HC)w + H^2,$$

where $\lambda = \Upsilon/\beta$, $\sigma = \kappa/\beta$, $C = \lambda - \sigma$ and $D = 2\lambda$. In Fig. 1 (*left*) a classification of the roots of the quartic equation $P(w)$ in the (σ, H) plane with $\lambda = 2$ is illustrated. The potential $P(w)$ and its corresponding phase portraits are shown in Fig. 1 (*right*) for $\sigma = 0.2$ and four different values of H, corresponding to each regions in the (σ, H) plane. In the physical region (A) of the (σ, H) plane, Eq. (3) has a solution $w(\xi)$ which is a periodic function, determined by the four real roots $w_0 < w_1 < w_2 < w_3$, which oscillates between the two lowest roots w_0 and w_1. The solution can be explicitly obtained by integrating Eq. (3).

$$w(\xi) = \frac{w_3 a \mathrm{sn}^2(r\xi|k) + w_0}{a \mathrm{sn}^2(r\xi|k) + 1}, \quad (4)$$

where $r = ND = \lambda\sqrt{(w_3 - w_1)(w_2 - w_0)}$ and $k = \sqrt{\frac{(w_3 - w_2)(w_1 - w_0)}{(w_3 - w_1)(w_2 - w_0)}}$, $a = \frac{w_1 - w_0}{w_3 - w_1}$.

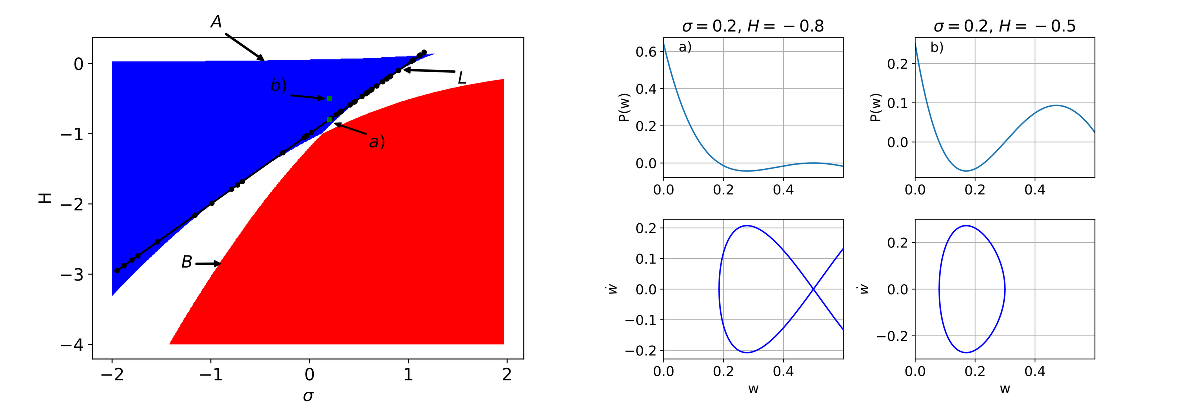

Figure: 1. (*left*)Regions of the (σ, H) plane in which the quartic equation $P(w) = 0$ has four real roots, region A (blue); two real roots (white region); and no real root region B (red). Large (black) dots on the line (L) indicates separatrices in the phase space, when two of the four roots are equal. (*right*) The potential and corresponding phase space for the four (green) squares at different regions in the (σ, H) plane (*left*), with σ and H according to the titles of the subplots.

Averaged equation for strong management

We consider the following form for the parameter of the cubic nonlinearity in the system (1) $\gamma(z) = \gamma_0 + \frac{1}{\varepsilon}\gamma_1\left(\frac{z}{\varepsilon}\right), \varepsilon \ll 1$, where $\gamma_1(\zeta + 1) = \gamma_1(\zeta)$, with $\zeta = z/\varepsilon$, is a periodic function. The following transformation to a new field for the FH exlude strong rapid varying terms from the GP equations. $u = \bar{u}e^{i\Gamma(z)|\bar{u}|^2}$, $v = \bar{v}$, where $\Gamma(z)$ is the anti-derivative of $\gamma_1(z)$, i.e., $\Gamma_z = \gamma_1(z)$. We obtain the following system of averaged coupled equations for the FH and SH

$$i\bar{u}_z - \frac{\gamma_1}{\omega}J_1\left(\frac{2\gamma_1}{\omega}|\bar{u}|^2\right)\bar{u}(\bar{u}^2\bar{v}^* + \bar{u}^{*2}\bar{v}) + \bar{u}^*\bar{v}J_0\left(\frac{2\gamma_1}{\omega}|\bar{u}|^2\right) + \gamma_0|\bar{u}|^2\bar{u} = 0,$$
$$i\bar{v}_z + q\bar{v} + \frac{\bar{u}^2}{2}J_0\left(\frac{2\gamma_1}{\omega}|\bar{u}|^2\right) = 0. \quad (5)$$

where $J_i(.), i = 0,1$ are the zero and first order Bessel functions. The Hamiltonian of the above averaged system, i.e. $i\bar{u}_z = -\frac{\partial H}{\partial \bar{u}^*}, i\bar{v}_z = -\frac{\partial H}{\partial \bar{v}^*}$, is then $H = \frac{\gamma_0}{2}|\bar{u}|^4 + q|\bar{v}|^2 + \frac{1}{2}J_0\left(\frac{2\gamma_1}{\omega}|\bar{u}|^2\right)(\bar{u}^2\bar{v}^* + \bar{u}^{*2}\bar{v})$. From the above equation we conclude the important result that the Hamiltonian have the same form as the standard $\chi^{(2)}$-system but with a renormalized effective quadratic nonlinearity parameter $\chi_{\text{eff}} = J_0\left(\frac{2\gamma_1}{\omega}|\bar{u}|^2\right)$. Note that the renormalization depends nonlinearly on the intensity of the FH field. For atomic-molecular BEC systems, it means that the renormalized atom-molecular interaction depend nonlinearly on the atomic population.

The validity of the averaging process is checked in Fig. 2 by solving the original equations (1) and the corresponding averaged equations (5).

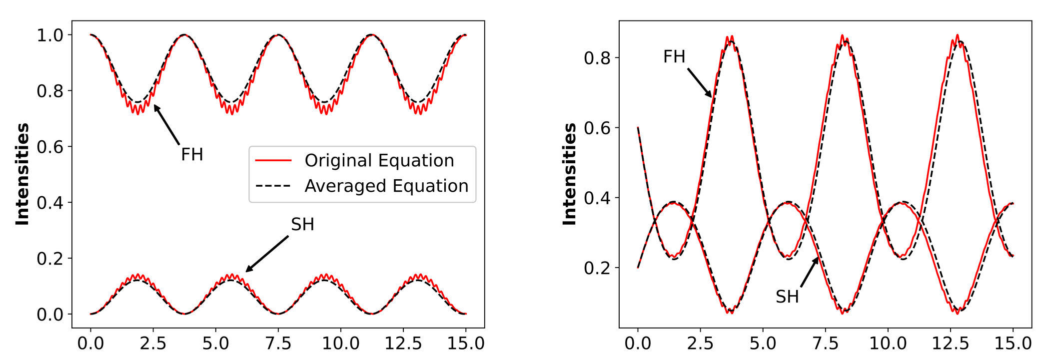

Figure: 2. Comparison of solutions from the original and the averaged equations. Numerical solutions of Eq. (1), solid (red) curves; and averaged from Eq. (5), dashed (black) curves. Parameters were here set to $\gamma_0 = 1$, $\gamma_1 = 20$, $\omega = 30$, and $q = 0.1$, with initial conditions $u(0) = \bar{u}(0) = 1$, $v(0) = \bar{v}(0) = 0$ (*left*) and $u(0) = \bar{u}(0) = \sqrt{0.6}\exp(i\phi)$, $v(0) = \bar{v}(0) = \sqrt{0.2}\exp(i\psi)$ (*right*) where $\phi = 0$ and $\psi = \frac{\pi}{2}$.

Evolution of cw under management with rapid and slow oscillations

We again apply conventional normalizations to the Eq. (5). We assume $\gamma = 0$ and obtain the differential equation:

$$\dot{w} = \sqrt{J_0^2(G(1-2w))(1-2w)^2w - 4\sigma^2w^2}. \quad (6)$$

where $w = \mu^2$, $G = 2\gamma_1 I/\omega$ and an overdot denotes differentiation with respect to $\xi = \beta Z$ Eq. (6) is valid when the following condition is fulfilled

$$|J_0(G(1-2w))(1-2w)| \geq |2\sigma\sqrt{w}|. \quad (7)$$

Fig. 3 shows the numerical results of integrating Eq. (6), i.e. the relative intensities, $w(z) = \mu^2 = |\bar{v}|^2/I$ and $1 - 2w(z) = \rho^2 = |\bar{u}|^2/I$. We further calculated numerically the percentage of the intensity of the SH wave, with respect to the total intensity, for different values of the parameters σ and G, see Fig. 4 (*left*), while Fig. 4 (*right*) illustrates the regions of parameters corresponding to $2\mu^2 \geq 0.8$ (region A), and $2\mu^2 < 0.8$ (region B). For the case weak management we choose parameters so that $k \ll 1$ and $a \ll 1$ in the solution (4), we can write Eq. (4) in the following approximative form

$$w(\xi) \approx w_3 a \mathrm{sin}^2(r\xi) = \frac{w_3 a}{2}(1 - \cos(2r\xi)). \quad (8)$$

To check the resonant behavior in the FH to SH oscillations, we take the frequency for the modulation of the cubic (Kerr) nonlinearity to be equal to the frequency of Eq. (8), i.e. $\omega = 2r$.

Results of the numerical integration are shown in Fig. 5.

To investigate possible chaotic regimes of oscillations, we calculate the Melnikov function [4] for the particular case $\gamma_0 = 0$ (corresponding to a periodic Feshbach resonance management close to zero in the atomic scattering length)

$$M(z_0) = -\frac{\pi\gamma_1}{3}\frac{\omega^2(2+\omega^2)}{\sinh(\frac{\pi\omega}{\sqrt{2}})}\sin(\omega z_0). \quad (9)$$

Since the Melnikov function above have infinite number of zeros, chaos in the harmonic generation is expected to occur. This is also found numerically, see Fig. 6.

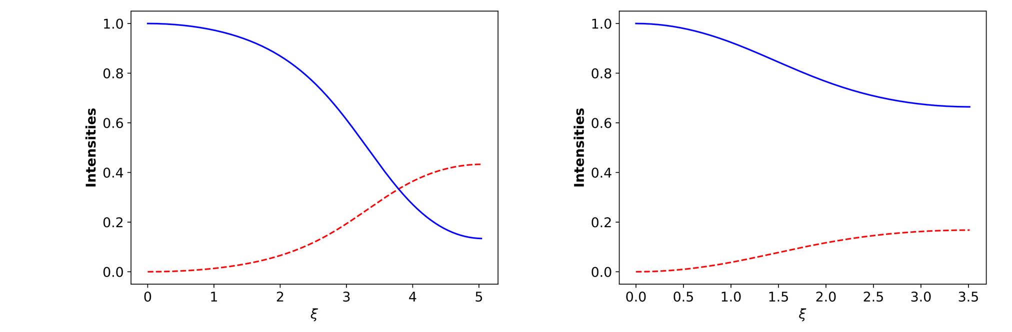

Figure: 3. Numerical evolution from Eq. (6) for the intensities corresponding to the case $G = 2$ (*left*) and $G = 4$ (*right*), with $\sigma = 0.1$ in both cases, and the initial conditions $\mu(0) = 0$, $\rho(0) = 1$. The solid (blue) curves shows the relative intensity of the FH wave (ρ^2), and the dashed (red) curves shows the relative intensity of the SH wave (μ^2). In the original equations these parameters corresponds to $\gamma_1/\omega = G/(2I)$ and $q = 2\sigma\sqrt{I}$ where $I = 1$.

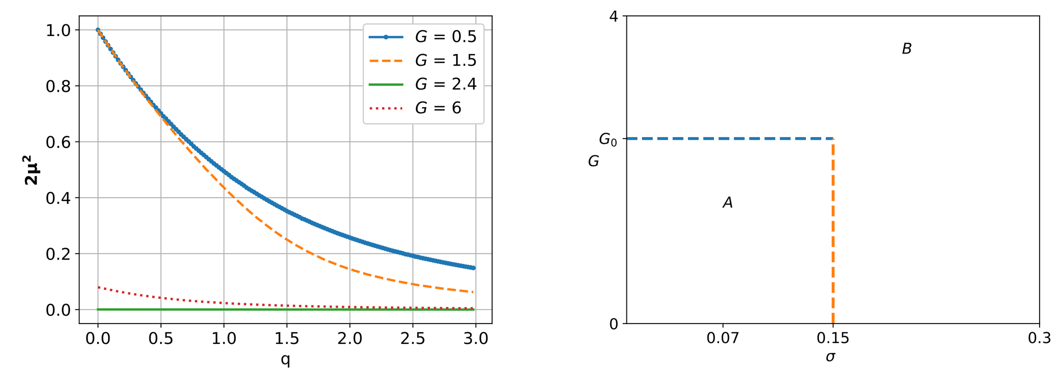

Figure: 4. Ratio of the intensity of the SH wave and the total intensity (*left*), as a function of q ($q = 2\sigma\sqrt{I}$), for different values of G ($G = 2\gamma_1 I/\omega$), see inset legend. Illustrative regions in the parameter plane (σ, G) obtained from the condition (7) (*right*), where $G_0 \simeq 2.405$ is the first zero of the Bessel's function J_0. More than 80% transformation of the energy defines region A, and less than 80% transformation defines region B.

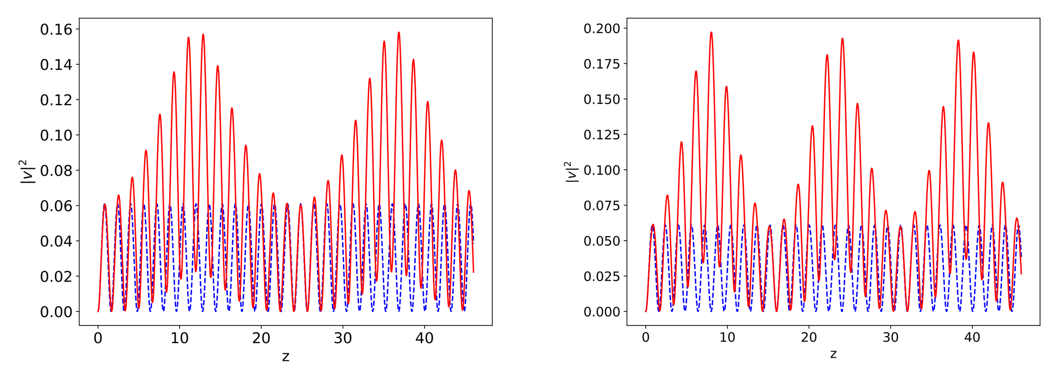

Figure: 5. Numerically calculated intensity of the SH wave for resonant modulations, from Eq. (1). Dashed (blue) curves shows oscillations of the SH wave in the case where the modulation is absent ($\gamma_1 = 0$). Solid (red) curves shows resonance behavior when the frequency (ω) of modulations for the cubic (Kerr) nonlinearity is equal to the frequency ($2r$) of oscillations for the SH wave without modulation. For the graph (*left*) $\gamma_1 = 0.2$, and for the graph (*right*) $\gamma_1 = 0.4$. The other parameters were $\omega = 2r \approx 3.95$, $\gamma_0 = 2$, $q = 0.2$, and $I = 1$ ($w_0 = 0.0$, $w_1 = 0.06090$, $w_2 = 0.7878$, $w_3 = 1.3011$).

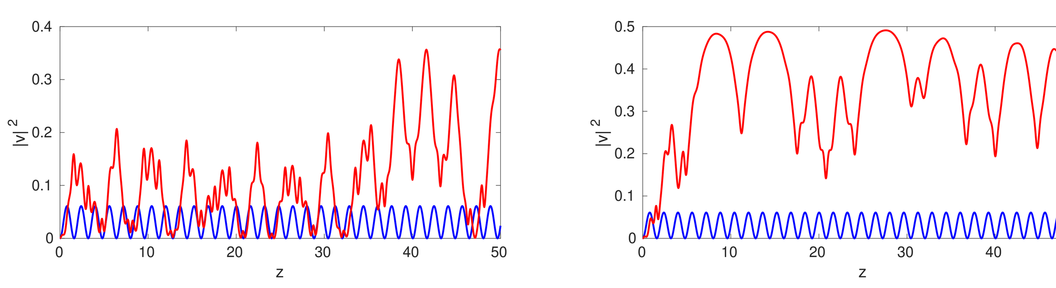

Figure: 6. Numerically calculated intensity of the SH wave for strong resonant modulations, from Eq. (1). For example, for the case of $\gamma_1 = 5$ (*left*) and $\gamma_1 = 6$ (*right*), chaos occurs. All the other parameters are the same as in Fig. 5.

Conclusion

To sum up, the obtained Hamiltonian for the averaged system shows that the result for rapid modulations of the Kerr nonlinearity leads to a nonlinear renormalization of the $\chi^{(2)}$ nonlinearity coefficient. In result, we have obtained the parameters of modulations for which the SH generation (association of atoms into a molecular condensate) can be suppressed dynamically. For the case of slow modulations, we find an enhancement of the SH generation (the molecular field) for the resonant value of the frequency for the modulations of the nonlinearity, which for strong amplitudes are chaotic. A sequential application of enhancing and suppressing modulations may be used in producing molecules from atoms.

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